ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 11, pp. 1053–1078.

On an Approach to Solving the Time-Optimization Problem for Linear Discrete-Time Systems Based on Krotov Method

D. N. Ibragimov[*,a] **and K. A. Tsarkov**[**,b]

[*] *Moscow Aviation Institute (National Research University), Moscow, Russia*
[**] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
e-mail: [a]*rikk.dan@gmail.com,* [b]*k6472@mail.ru*

Received January 30, 2024
Revised September 13, 2024
Accepted September 20, 2024

Abstract—In the paper, we develop a method for studying the time-optimization problem for a linear system with discrete time, which allows, in the general case, to improve known upper estimates of the objective function and to find guaranteeing control processes. We obtain sufficient conditions for convergence to the optimal solution in the problem and implement the method as an effective numerical algorithm.

Keywords: time-optimization problem, linear systems, discrete time, sequential global improvement

DOI: 10.31857/S0005117924110019

1. INTRODUCTION

The time-optimization problem is well-known as an optimal control problem, in which the objective function is the time spent by the system to reach some given terminal state [1–3]. For systems with continuous time, this problem naturally fits into the general problem of classical optimal control theory. In the case of linear continuous-time systems, the application of Pontryagin maximum principle [1] guarantees that any solution to the problem has the form of a relay control function.

Systems with discrete time have a number of fundamental differences in such a case [4–6]. While most of the problems in the discrete-time optimal control theory can be solved by the discrete maximum principle [6, 7] and/or the dynamic programming method [8], these approaches are totally unapplicable to solving the time-optimization problem even for a linear system. The main reasons here are the irregularity of the extremum for almost all initial states, the non-uniqueness of the optimal trajectory, and the discrete nature of the objective function [9, 10]. The use of many modern results in the theory [11, 12] in relation to this problem also turns out to be incorrect, and the known papers discussing the time-optimization problem for discrete-time systems cover only a number of special cases [13, 14]. From a practical point of view, it is important to obtain results that can be used in the case of a linear system of arbitrary dimension with a convex set of geometric constraints on control. When considering such systems, the results of the above-mentioned papers are either difficult to implement in computational terms or are only applicable under significant additional assumptions.

In this paper, we develop an efficient numerical algorithm for constructing a time-optimal control for linear discrete-time systems. One of the most well-known and justified numerical schemes for solving various linear optimal control problems is Krotov method [15, 16]. It is based on the sufficient conditions for global optimality [17] and the principle of extending optimization problems [18] developed by V.F. Krotov, V.I. Gurman and M.M. Khrustalev. A number of papers are also devoted to the implementation of this method in the case of discrete and discrete-continuous in time control systems, see [19, 20]. The Krotov method is an iterative procedure of constructing sequential improvements of some pre-selected control of a given dynamical system. The most important feature of the method is its non-locality. This means that after each iteration, new controls could not be close to those found in the previous steps in the sense of any distance in the space of admissible controls or in the sense of the values of an objective function. In the case of linear systems, this feature appears most clearly, since it is often possible to determine the optimal control already in the first iteration of Krotov method from any initial approximation [19, 21].

In this paper, Krotov method is applied to find an optimal-time control for known estimates of the optimal time value. Several alternative approaches are proposed for constructing the estimates. In general, the results of [9, 10, 22] can be used for these purposes, although in many situations their computational complexity is significant. Therefore, in Section 3 we propose a new approach to constructing optimal time estimates for the case when the matrix of the considered linear system is diagonalizable. After estimates have been constructed, it is possible to proceed to a problem with a fixed time. This reduction is described in detail in Section 4. Then Krotov method is applied to the resulting problem (Section 5). In Section 6 a general numerical algorithm for studying the time-optimization problem is formulated. Section 7 contains a number of examples illustrating the efficiency and application features of the algorithm in solving specific problems.

In comparison with the previous works of one of the authors [9, 10, 22], it is not assumed to obtain analytical conditions for optimality of a control process in the time-optimization problem. Instead, we present a numerical procedure that allows, in some cases, to approximately find time-optimal processes. In comparison with [22], where two-sided estimates of the optimal time were constructed with geometric methods, the estimates in this article are constructed analytically, but for a smaller class of systems. The idea of using Krotov's global improvement method in studying the time-optimization problem in discrete time is completely new. Before this, Krotov method was used by one of the authors in studying some problems of optimal control for continuous-time systems [23].

In this paper, we restrict ourselves to considering stationary linear discrete-time systems with a non-singular matrix and a convex compact set of geometric constraints on control. The non-singularity condition is used to prove the convergence of the proposed iterative procedure. The stationarity condition is unimportant and is assumed for simplicity.

2. PROBLEM STATEMENT AND GENERAL IDEA OF THE SOLUTION

Consider a linear stationary system with discrete time

$$x(k+1) = Ax(k) + u(k), \quad k \in \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system, $u(k) \in U$ is the control, U is a convex compact set in $\mathbb{R}^n$ such that $0 \in \mathrm{int}U$, $A \in \mathbb{R}^{n\times n}$ is a given non-singular matrix ($\det A \neq 0$). The initial condition for the system (1) is fixed:

$$x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

It is required to calculate the minimum number of steps $N_{\min}$, in which it is possible to transfer the system (1) from a given initial state x_0 to the origin and to construct an optimal process $\{x^*(k), u^*(k-1)\}_{k=1}^{N_{\min}}$, satisfying the condition $x^*(N_{\min}) = 0$. The number $N_{\min}$ will be further called the optimal time of the system (1) with the initial condition (2) and we will assume that the considered problem is solvable, i.e. $N_{\min} < \infty$.

The problem will be investigated in two separate stages. Let us first give a description of them.

At the first stage, we estimate the optimal time $N_{\min}$. In some situations, $N_{\min}$ can be calculated exactly, but in the general case, we assume to find a two-sided estimate

$$\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}, \tag{3}$$

where the equality $\overline{N_{\min}} = \underline{N_{\min}}$ is not excluded. For this purpose, theoretical results from [9, 10] and algorithmic approaches from [22] can be used. In the next section, we propose a new approach to constructing the dual estimates in the case where the matrix A of the system (1) has n linearly independent eigenvectors.

At the second stage, we solve optimal control problems for fixed operation times N for the system (1)–(2) with respect to the functional $\|x(N)\|^2$, the squared Euclidean norm of the vector $x(N)$, where N takes the values $\overline{N_{\min}}, \ldots, \underline{N_{\min}}$. The smallest N for which the minimum value $\|x^*(N)\|^2$ is zero gives the optimal time $N_{\min} = N$, and the corresponding $\{x^*(k), u^*(k-1)\}_{k=1}^{N}$ is the optimal process. The method for finding the optimal processes is formulated and justified in Sections 4 and 5. Section 6 is devoted to the joint algorithmic implementation of the described stages.

3. OPTIMAL TIME ESTIMATE

As demonstrated in [9, 10, 22], the calculation of $N_{\min}$ can be reduced to constructing a class of null-controllable sets $\{\Xi(N)\}_{N=0}^{\infty}$. Here $\Xi(N) \subset \mathbb{R}^n$ is the set of those initial states from which the system (1) can be transferred to the origin in N steps, i.e.

$$\Xi(N) := \begin{cases} \{\xi \in \mathbb{R}^n \mid \exists u(0), \ldots, u(N-1) \in U \colon x(N) = 0\}, & N \in \mathbb{N}, \\ \{0\}, & N = 0, \end{cases} \tag{4}$$

where $x(N)$ denotes the solution of the system (1) at $x(0) = \xi$.

Since the time-optimization problem for a given initial state (2) is assumed to be solvable, then the following inclusion holds:

$$x_0 \in \bigcup_{N=0}^{\infty} \Xi(N).$$

Therefore, taking into account (4), we have

$$N_{\min} = \min\{N \in \mathbb{N} \cup \{0\} \colon x_0 \in \Xi(N)\}. \tag{5}$$

The procedure of constructing $\{\Xi(N)\}_{N=0}^{\infty}$ is very complex, which is due to the following representation of 0-controllability sets.

Lemma 1 [9, Lemma 1]. *Let the sequence $\{\Xi(N)\}_{N=0}^{\infty}$ be defined according to* (4) *and* $\det A \neq 0$. *Then for all $N \in \mathbb{N}$ the relation*

$$\Xi(N) = -\sum_{k=1}^{N} \left(A^{-k} U\right)$$

holds, where the sum symbol denotes the Minkowski sum of sets.

Minkowski sum of convex sets is generally computationally intractable. For example, let U be a polytope in $\mathbb{R}^n$. Then every set $\Xi(N)$ is also a polytope [24, Corollary 19.3.2] and the descriptive complexity of the polytopes $\Xi(N)$ (i.e., the number of their vertices) grows exponentially in N [25, Theorem 4.1.2].

However, under some additional assumptions on the matrix A and the set U, it is possible to compute a two-sided a priori estimate of $N_{\min}$ without having to construct the sequence $\{\Xi(N)\}_{N=0}^{\infty}$ explicitly. One such case is considered below.

Let us introduce some auxiliary notations. Let $u_{\max} > 0$ and $\lambda \neq 0$ be some real numbers. Consider the mapping $F(\cdot; u_{\max}, \lambda)\colon \mathbb{R} \to [0; +\infty)$, defined in the form

$$F(\alpha; u_{\max}, \lambda) = \begin{cases} \dfrac{|\alpha|}{u_{\max}}, & |\lambda| = 1, \\ -\dfrac{\ln\left(1 - \frac{|\alpha|}{u_{\max}}(|\lambda| - 1)\right)}{\ln|\lambda|}, & |\lambda| \neq 1. \end{cases} \tag{6}$$

For an arbitrary $\varphi \in \mathbb{R}$, we denote by $A_\varphi \in \mathbb{R}^{2\times 2}$ the rotation matrix

$$A_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

and we denote by $\mathcal{B}_R$ the closed ball of radius R centered at origin in $\mathbb{R}^2$. We also introduce a notation for the m-ary Cartesian product of arbitrary sets $V_1, \ldots, V_m$:

$$\bigotimes_{i=1}^{m} V_i := V_1 \times \ldots \times V_m.$$

Lemma 2. *Let the condition $N_{\min} < \infty$ be satisfied in the system (1), there exist numbers $\lambda_1, \ldots, \lambda_{n_1} \neq 0$, $r_1, \ldots, r_{n_2} > 0$, $\varphi_1, \ldots, \varphi_{n_2} \in \mathbb{R}$ such that*

$$A = \begin{pmatrix} \lambda_1 & & & \ldots & & 0 \\ & \ddots & & & & \\ & & \lambda_{n_1} & & & \\ \vdots & & & r_1 A_{\varphi_1} & & \vdots \\ & & & & \ddots & \\ 0 & & & \ldots & & r_{n_2} A_{\varphi_{n_2}} \end{pmatrix},$$

and numbers $u_{1,\max}, \ldots, u_{n_1,\max}, R_{1,\max}, \ldots, R_{n_2,\max} > 0$ satisfying the condition

$$U = \bigotimes_{i=1}^{n_1} [-u_{i,\max}; u_{i,\max}] \times \bigotimes_{j=1}^{n_2} \mathcal{B}_{R_{j,\max}},$$

where $n_1, n_2 \geqslant 0$ and $n_1 + 2n_2 = n$.

Then the inclusion $x_0 = (x_{0,1}, \ldots, x_{0,n})^{\mathrm{T}} \in \Xi(N)$ holds if and only if the inequality

$$N \geqslant \max\left\{ \max_{i=\overline{1,n_1}} F(x_{0,i}; u_{i,\max}, \lambda_i);\ \max_{j=\overline{1,n_2}} F\left(\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2}; R_{j,\max}, r_j\right) \right\}$$

is valid.

For the convenience of the reader, the proof of the statements in this and subsequent sections is transferred to the Appendix.

Note that within Lemma 2 the values $n_2 = 0$ or $n_1 = 0$ are admissible. In this case, the matrix A does not have the corresponding blocks, and all its eigenvalues are either real or essentially complex.

Corollary 1. *Under the assumptions of Lemma 2, due to (5), the exact equality holds*

$$N_{\min} = \left\lceil \max\left\{ \max_{i=\overline{1,n_1}} F(x_{0,i}; u_{i,\max}, \lambda_i);\ \max_{j=\overline{1,n_2}} F\left(\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2}; R_{j,\max}, r_j\right) \right\} \right\rceil,$$

where $\lceil\alpha\rceil$ means the minimum integer not less than α:

$$\lceil\alpha\rceil := \min\{k \in \mathbb{Z}\colon \alpha \leqslant k\}, \quad \alpha \in \mathbb{R}.$$

The result of Corollary 1 is applicable only to a small class of systems (A, U) described by the conditions of Lemma 2. But it can be used to obtain dual estimates of the optimal time in the case of an arbitrary diagonalizable matrix A and a convex set U. For this purpose, we should consider auxiliary systems of the form (1) satisfying the conditions of Lemma 2 and use the following statement.

Lemma 3. *Let the inclusion* $\underline{U} \subset U \subset \overline{U}$ *hold, where* $\underline{U}$, U, $\overline{U} \subset \mathbb{R}^n$ *are convex and compact sets containing* 0*, the time-optimization problem for* $x_0 \in \mathbb{R}^n$ *is solvable for systems* $(A, \underline{U})$, (A, U), $(A, \overline{U})$*, and* $\underline{N_{\min}}$, $N_{\min}$, $\overline{N_{\min}}$ *are optimal time values in the time-optimization problem for these systems, respectively. Then*

$$\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}.$$

It is known that any matrix $A \in \mathbb{R}^{n\times n}$ with n linearly independent eigenvectors can be reduced to the form presented in Lemma 2 by the spectral decomposition [26, Theorem 3.4.5]. The columns of the matrix of this transformation $S \in \mathbb{R}^{n\times n}$ are either eigenvectors for real eigenvalues or their imaginary and real parts for complex eigenvalues. A similar linear transformation can be applied to the entire system (A, U), passing to the equivalent system $(S^{-1}AS, S^{-1}U)$ as demonstrated in [27]. The following result holds.

Lemma 4 [27, Lemma 2]. *Let* $S \in \mathbb{R}^{n\times n}$, $\det S \neq 0$, (A, U) *be a system of the form* (1)*, and* $\{\tilde{\Xi}(N)\}_{N=0}^{\infty}$ *denote the class of null-controllable sets of the system* $(S^{-1}AS, S^{-1}U)$*. Then*

$$\Xi(N) = S\tilde{\Xi}(N), \quad N \in \mathbb{N} \cup \{0\}.$$

In the context of the considered problem, the set $S^{-1}U$ can be estimated from above and below by the sets $\underline{U}, \overline{U} \subset \mathbb{R}^n$ satisfying the conditions of Lemma 2. In combination with Lemma 3, this leads to the desired estimates of $N_{\min}$ in the original time-optimization problem. More precisely, the following statement holds.

Theorem 1. *Let in the system* (1) *for a given initial state* $x_0 \in \mathbb{R}^n$ *the condition* $N_{\min} < \infty$ *hold, the matrix* $A \in \mathbb{R}^{n\times n}$ *have* n *linearly independent eigenvectors,* $\det A \neq 0$, $S \in \mathbb{R}^{n\times n}$ *be the transition matrix to the real Jordan basis of the matrix* A*:*

$$S^{-1}AS = \Lambda = \begin{pmatrix} \lambda_1 & & & \dots & & 0 \\ & \ddots & & & & \\ & & \lambda_{n_1} & & & \\ \vdots & & & r_1 A_{\varphi_1} & & \vdots \\ & & & & \ddots & \\ 0 & & & \dots & & r_{n_2} A_{\varphi_{n_2}} \end{pmatrix}.$$

Then the following estimate of $N_{\min}$ *is valid:*

$$\left\lceil \max\left\{ \max_{i=\overline{1,n_1}} F(y_{0,i}; u''_{i,\max}, \lambda_i);\ \max_{j=\overline{1,n_2}} F\left(\sqrt{y_{0,n_1+2j-1}^2 + y_{0,n_1+2j}^2}; R''_{j,\max}, r_j\right)\right\}\right\rceil \leqslant N_{\min}$$
$$\leqslant \left\lceil \max\left\{ \max_{i=\overline{1,n_1}} F(y_{0,i}; u'_{i,\max}, \lambda_i);\ \max_{j=\overline{1,n_2}} F\left(\sqrt{y_{0,n_1+2j-1}^2 + y_{0,n_1+2j}^2}; R'_{j,\max}, r_j\right)\right\}\right\rceil,$$

where $y_0 = S^{-1}x_0$*, and the numbers* $u'_{i,\max}$, $u''_{i,\max}$, $R'_{j,\max}$, $R''_{j,\max} > 0$, $i = \overline{1, n_1}$, $j = \overline{1, n_2}$*, are determined by the condition*

$$\bigotimes_{i=1}^{n_1} [-u'_{i,\max}; u'_{i,\max}] \times \bigotimes_{j=1}^{n_2} \mathcal{B}_{R'_{j,\max}} \subset S^{-1}U \subset \bigotimes_{i=1}^{n_1} [-u''_{i,\max}; u''_{i,\max}] \times \bigotimes_{j=1}^{n_2} \mathcal{B}_{R''_{j,\max}}.$$

Theorem 1 allows us to obtain a priori estimates of the value $N_{\min}$ for the initial state x_0 only in the case when the system matrix A has n linearly independent eigenvectors. This fact is essential, since results similar to Lemma 2 cannot be obtained if the real Jordan form of the matrix A contains Jordan cells corresponding to multiple eigenvalues. This is due to the complexity of constructing sets invariant with respect to this kind of linear transformation.

Remark 1. The statement of Theorem 1 involves the $u'_{i,\max}$, $u''_{i,\max}$, $R'_{j,\max}$, $R''_{j,\max}$ values, which determine the two-sided estimate of the optimal time $N_{\min}$. According to Lemma 3, the greatest accuracy of the lower estimate will be achieved with the minimum admissible values of $u''_{i,\max}$, $R''_{j,\max}$, which can be calculated in the course of solving the following convex programming problems:

$$u''_{i,\max} = \max_{u \in S^{-1}U} |u_i|, \quad i = \overline{1, n_1}, \tag{7}$$

$$R''_{j,\max} = \max_{u \in S^{-1}U} \sqrt{u^2_{n_1+2j-1} + u^2_{n_1+2j}}, \quad j = \overline{1, n_2}. \tag{8}$$

Determining the best values of the parameters $u'_{1,\max}, \ldots, u'_{n_1,\max}, R'_{1,\max}, \ldots, R'_{n_2,\max}$ for the upper bound is a much more complex problem. This is due to the need to solve a minimax problem that depends on the initial state x_0. However, if a certain set of admissible values of these parameters is known, for which the inclusion

$$\underline{U} = \bigotimes_{i=1}^{n_1} [-u'_{i,\max}; u'_{i,\max}] \times \bigotimes_{j=1}^{n_2} \mathcal{B}_{R'_{j,\max}} \subset S^{-1}U$$

holds, then all parameters except one can be fixed, and the last one can be selected as a result of solving one of the following two optimization problems:

$$u'_{i_0,\max} = \max \left\{ u > 0 \colon \bigotimes_{i=1}^{i_0-1} [-u'_{i,\max}; u'_{i,\max}] \times [-u; u] \times \bigotimes_{i=i_0+1}^{n_1} [-u'_{i,\max}; u'_{i,\max}] \right.$$
$$\left. \times \bigotimes_{j=1}^{n_2} \mathcal{B}_{R'_{j,\max}} \subset S^{-1}U \right\}, \quad i_0 = \overline{1, n_1}, \tag{9}$$

$$R'_{j_0,\max} = \max \left\{ R > 0 \colon \bigotimes_{i=1}^{n_1} [-u'_{i,\max}; u'_{i,\max}] \times \bigotimes_{j=1}^{j_0-1} \mathcal{B}_{R'_{j,\max}} \times \mathcal{B}_R \right.$$
$$\left. \times \bigotimes_{j=j_0+1}^{n_2} \mathcal{B}_{R'_{j,\max}} \subset S^{-1}U \right\}, \quad j_0 = \overline{1, n_2}. \tag{10}$$

4. FIXED-TIME PROBLEM

Let for some integer N the two-sided estimate $\overline{N_{\min}} \leqslant N \leqslant \underline{N_{\min}}$ be satisfied, where the values of $\overline{N_{\min}} \leqslant \underline{N_{\min}}$ are obtained based on the results from [9, 10, 22] or the previous section. Note that the method of constructing the estimates is not essential here, since only their numerical values will be used in further discussions.

We introduce the notation $\mathcal{U} := \{k \mapsto u(k) : \{0, 1, \ldots, N-1\} \to U\}$, $\mathcal{X} := \{k \mapsto x(k) : \{0, 1, \ldots, N\} \to \mathbb{R}^n \mid x(0) = x_0\}$ and consider the problem

$$J(x(N)) = \|x(N)\|^2 \to \min_{u \in \mathcal{U}}. \tag{11}$$

If $N_{\min}$ is the optimal time for (1)–(2), then the minimum in (11) for $N \geqslant N_{\min}$ is achieved and is equal to zero. If $N = N_{\min}$, then all solutions of (11) are solutions (optimal controls generating optimal processes) in the original time-optimization problem for system (1) with the initial condition (2).

It is known [9, 28, 29] that optimal controls in (11) in the case of $N \geqslant N_{\min}$ are special in the sense that the necessary optimality conditions in the form of discrete maximum principle turn out to be meaningless. Various regularization methods proposed in [9, 10, 15, 18, 28, 29] allow to solve this problem, but lead, as a rule, to significant computational difficulties. Therefore, we will approach the solution of (11) from the position of constructing a globally minimizing sequence. We will proceed as follows.

Let some (non-optimal) control $\hat{u} \in \mathcal{U}$ be already given, this control corresponds to a trajectory $\hat{x} \in \mathcal{X}$, which is a solution to the system of recurrence relations (1) for $u = \hat{u}$ with the initial condition (2), and the quality of this control $\hat{u}$ is numerically characterized by the value $J(\hat{x}(N))$. Under these assumptions, we construct a new control $\tilde{u} \in \mathcal{U}$, to which a new trajectory $\tilde{x} \in \mathcal{X}$ corresponds and for which the quality of the control $J(\tilde{x}(N))$ satisfies the inequality

$$J(\tilde{x}(N)) < J(\hat{x}(N)). \tag{12}$$

If this can be done and $J(\tilde{x}(N)) > 0$, then we redesignate $\tilde{u}$ by $\hat{u}$ and repeat the procedure. To construct an improvement $\tilde{u}$ for a given control $\hat{u}$, we will use the method proposed by V.F. Krotov and then developed in various directions in numerous works by his students and followers (see, for example, [16, 18–21, 23, 30]).

In [16, 19, 21] it was noted that in the case of systems linear in state and linear terminal quality functional, Krotov's method demonstrates the highest rate of improvement. Moreover, in this case the procedure of constructing improvements is significantly simplified, since it turns out to be possible to apply the simplest, linear, implementation of the method. In order to use these advantages, we will first consider the regularization transformation, which allows us to reduce the problem (11) with respect to the system (1) and the initial condition (2) to an equivalent problem with a functional linear in state.

For $x \in \mathbb{R}^n$ we introduce the notation $X := xx^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. Then for any k we have

$$\begin{aligned} X(k+1) &:= x(k+1)x(k+1)^{\mathrm{T}} = (Ax(k)+u(k))(Ax(k)+u(k))^{\mathrm{T}} \\ &= AX(k)A^{\mathrm{T}} + Ax(k)u(k)^{\mathrm{T}} + u(k)x(k)^{\mathrm{T}}A(k)^{\mathrm{T}} + u(k)u(k)^{\mathrm{T}} \end{aligned} \tag{13}$$

and $X(0) = x_0x_0^{\mathrm{T}}$. Let's consider the problem

$$\mathcal{J}(X(N)) = \operatorname{tr}[X(N)] \to \min_{u\in\mathcal{U}}. \tag{14}$$

It is clear that $\mathcal{J}(X(N)) = \mathcal{J}(x(N)x(N)^{\mathrm{T}}) = J(x(N))$, and problem (14) is equivalent to problem (11).

Let us recall the construction of classical necessary optimality conditions in the problem (14) (see, for example, [31]). Let us compose the Hamilton–Pontryagin function

$$H(x, X, \psi, \Psi, u) = \langle \psi, Ax+u\rangle + \operatorname{tr}\left[\Psi\left(AXA^{\mathrm{T}} + Axu^{\mathrm{T}} + ux^{\mathrm{T}}A^{\mathrm{T}} + uu^{\mathrm{T}}\right)\right]$$

and the system of dual equations

$$\begin{gathered} \psi(k) = A^{\mathrm{T}}\psi(k+1) + 2A^{\mathrm{T}}\Psi(k+1)u(k), \quad \psi(N) = 0, \\ \Psi(k) = A^{\mathrm{T}}\Psi(k+1)A, \quad \Psi(N) = -I, \end{gathered}$$

where $\psi \in \mathbb{R}^n$, $\Psi \in \mathbb{R}^{n\times n}$, I is the identity matrix of dimensions $n \times n$. If the control $\hat{u} \in \mathcal{U}$ is optimal in problem (14), then the following relations of the discrete vector-matrix maximum principle are satisfied:

$$H(\hat{x}(k),\hat{X}(k),\hat{\psi}(k+1),\hat{\Psi}(k+1),\hat{u}(k)) = \max_{v\in U} H(\hat{x}(k),\hat{X}(k),\hat{\psi}(k+1),\hat{\Psi}(k+1),v), \tag{15}$$

$$\hat{x}(k+1) = A\hat{x}(k) + \hat{u}(k), \quad \hat{x}(0) = x_0, \tag{16}$$

$$\hat{X}(k+1) = A\hat{X}(k)A^{\mathrm{T}} + A\hat{x}(k)\hat{u}(k)^{\mathrm{T}} + \hat{u}(k)\hat{x}(k)^{\mathrm{T}}A^{\mathrm{T}} + \hat{u}(k)\hat{u}(k)^{\mathrm{T}}, \quad \hat{X}(0) = x_0x_0^{\mathrm{T}}, \tag{17}$$

$$\hat{\psi}(k) = A^{\mathrm{T}}\hat{\psi}(k+1) + 2A^{\mathrm{T}}\hat{\Psi}(k+1)\hat{u}(k), \quad \hat{\psi}(N) = 0, \tag{18}$$

$$\hat{\Psi}(k) = A^{\mathrm{T}}\hat{\Psi}(k+1)A, \quad \hat{\Psi}(N) = -I, \tag{19}$$

where k takes all possible values from the set $\{0,1,\ldots,N-1\}$.

In the relations (15)–(19) the matrices $\hat{X}(k)$ and $\hat{\Psi}(k)$ have the role of auxiliary regularization variables and can be removed from consideration in the future. Indeed, due to (19) we have

$$\hat{\Psi}(k) = -(A^{\mathrm{T}})^{N-k}A^{N-k} = -(A^{N-k})^{\mathrm{T}}A^{N-k},$$

and for any k the following equality holds:

$$\begin{aligned} &H(\hat{x}(k),\hat{X}(k),\hat{\psi}(k+1),\hat{\Psi}(k+1),v) \\ &= \langle\hat{\psi}(k+1), A\hat{x}(k)+v\rangle + \mathrm{tr}[\hat{\Psi}(k+1)(A\hat{X}(k)A^{\mathrm{T}} + A\hat{x}(k)v^{\mathrm{T}} + v\hat{x}(k)^{\mathrm{T}}A^{\mathrm{T}} + vv^{\mathrm{T}})] \\ &= \langle\hat{\psi}(k+1) + 2\hat{\Psi}(k+1)A\hat{x}(k), v\rangle + \langle v, \hat{\Psi}(k+1)v\rangle + \hat{H}_0(k) \\ &= \langle\hat{\psi}(k+1) - 2(A^{N-k-1})^{\mathrm{T}}A^{N-k}\hat{x}(k), v\rangle - \langle v, (A^{N-k-1})^{\mathrm{T}}A^{N-k-1}v\rangle + \hat{H}_0(k), \end{aligned}$$

where $\hat{H}_0(k) = \langle\hat{\psi}(k+1), A\hat{x}(k)\rangle + \mathrm{tr}[\hat{\Psi}(k+1)A\hat{X}(k)A^{\mathrm{T}}]$ does not depend on v.

Therefore, the system of relations of the discrete maximum principle (15)–(19) is equivalent to the system

$$\begin{aligned} &\langle\hat{\psi}(k+1) - 2(A^{N-k-1})^{\mathrm{T}}A^{N-k}\hat{x}(k), \hat{u}(k)\rangle - \langle\hat{u}(k), (A^{N-k-1})^{\mathrm{T}}A^{N-k-1}\hat{u}(k)\rangle \\ &= \max_{v\in U}\Big(\langle\hat{\psi}(k+1) - 2(A^{N-k-1})^{\mathrm{T}}A^{N-k}\hat{x}(k), v\rangle - \langle v, (A^{N-k-1})^{\mathrm{T}}A^{N-k-1}v\rangle\Big), \end{aligned} \tag{20}$$

$$\hat{x}(k+1) = A\hat{x}(k) + \hat{u}(k), \quad \hat{x}(0) = x_0, \tag{21}$$

$$\hat{\psi}(k) = A^{\mathrm{T}}\hat{\psi}(k+1) - 2(A^{N-k})^{\mathrm{T}}A^{N-k-1}\hat{u}(k), \quad \hat{\psi}(N) = 0. \tag{22}$$

It is important to note that this regularized system is not computationally equivalent to the degenerate system of relations of the discrete maximum principle for the problem (11) and theoretically allows to find optimal controls in the original problem. However, as has already been said, determining the optimal control directly from the conditions (20)–(22) is associated with serious computational difficulties. Therefore, in accordance with the above approach, we will seek improvements $\tilde{u} \in \mathcal{U}$ of the given (non-optimal) control $\hat{u} \in \mathcal{U}$ in the sense of the inequality (12) using the obtained regular constructions.

5. KROTOV METHOD

Let $\hat{u} \in \mathcal{U}$ be an arbitrary control, $\hat{x} \in \mathcal{X}$ satisfies the equation (21), and $\hat{\psi} \in \mathcal{X}'$ satisfies the dual equation (22). Here $\mathcal{X}' := \{k \mapsto \psi(k) : \{0,1,\ldots,N\} \to \mathbb{R}^n \mid \psi(N) = 0\}$. Consider the function $\hat{\varphi} : \{0,\ldots,N\} \times \mathbb{R}^n \to \mathbb{R}$ of the form

$$\hat{\varphi}(k,x) = \langle\hat{\psi}(k), x\rangle - \|A^{N-k}x\|^2.$$

For $k \in \{0, \dots, N-1\}$ and $x, u \in \mathbb{R}^n$ we introduce the following notations, consistent with [15, 16]:

$$\hat{R}(k, x, u) = \hat{\varphi}(k+1, Ax+u) - \hat{\varphi}(k, x),$$
$$\hat{G}(x) = \hat{\varphi}(N, x) - \hat{\varphi}(0, x_0) + J(x).$$

Here the choice of notations $\hat{\varphi}, \hat{R}, \hat{G}$ is related to the fact that these functions are determined by the element $\hat{\psi} \in \mathcal{X}'$, i.e., ultimately, by an arbitrarily chosen control $\hat{u} \in \mathcal{U}$. It is clear that by construction for any values of k, x and u we have

$$\hat{R}(k, x, u) = H(x, xx^{\mathrm{T}}, \hat{\psi}(k+1), -(A^{N-k-1})^{\mathrm{T}} A^{N-k-1}, u) - \hat{\varphi}(k, x),$$
$$\hat{G}(x) \equiv -\langle \hat{\psi}(0), x_0 \rangle + \|A^N x_0\|^2.$$

The following result is the main statement on the improvement. Improvement theorems were first formulated by V.F. Krotov (see, for example, [16, Theorem 1]).

Theorem 2. *Let $\hat{u} \in \mathcal{U}$, $\hat{x} \in \mathcal{X}$, $\hat{\psi} \in \mathcal{X}'$ satisfy the relations* (21) *and* (22). *Let $\tilde{u} \in \mathcal{U}$ satisfy the condition*

$$\hat{R}(k, \tilde{x}(k), \tilde{u}(k)) = \max_{v \in U} \hat{R}(k, \tilde{x}(k), v) \quad \forall k \in \{0, \dots, N-1\}, \tag{23}$$

where $\tilde{x}(0) = x_0$ and for $k = 0, \dots, N-1$

$$\tilde{x}(k+1) = A\tilde{x}(k) + \tilde{u}(k). \tag{24}$$

Then there is a non-strict improvement in the problem (12), *i.e.*

$$J(\tilde{x}(N)) \leqslant J(\hat{x}(N)).$$

Remark 2. Let $\hat{u}$ and $\hat{x}$ be taken from Theorem 2. Then the pair $(\hat{x}, \hat{u})$ satisfies the relations of the discrete maximum principle (20)–(22) if and only if for $\tilde{u} = \hat{u}$ the condition (23) is satisfied for $\tilde{x} = \hat{x}$, i.e.

$$\hat{R}(k, \hat{x}(k), \hat{u}(k)) = \max_{v \in U} \hat{R}(k, \hat{x}(k), v), \quad k = 0, \dots, N-1.$$

Remark 3. The condition (23) is equivalent to the condition

$$\tilde{u}(k) \in \operatorname{Arg}\max_{v \in U} \left(\langle \hat{\psi}(k+1) - 2(A^{N-k-1})^{\mathrm{T}} A^{N-k} \tilde{x}(k), v \rangle - \|A^{N-k-1} v\|^2 \right).$$

As an elementary corollary, we note how the result of Theorem 2 is related to solutions of extremal problems (11) and (14).

Corollary 2. *Let $\hat{u} \in \mathcal{U}$ be an optimal control in problems* (11) *and* (14), *and let $\hat{x} \in \mathcal{X}$ and $\hat{\psi} \in \mathcal{X}'$ satisfy the relations* (21), (22). *Then for any $\tilde{u} \in \mathcal{U}$ satisfying the condition* (23), *the equality $J(\tilde{x}(N)) = J(\hat{x}(N))$ holds.*

The statement of Theorem 2 about the fulfillment of non-strict inequality $J(\tilde{x}(N)) \leqslant J(\hat{x}(N))$ remains valid even if the condition $\det A \neq 0$ in the original problem is not satisfied. But for systems of the form (1) with a non-singular matrix A, it is possible to establish a closer connection between unimprovability in the sense of the inequality (12) in Theorem 2 and extremals in (14).

Theorem 3. *Let A be a non-singular matrix in the system* (1), *$\hat{u}$ and $\hat{x}$ be taken from Theorem 2. Then there exists a unique pair $(\tilde{x}, \tilde{u})$ satisfying conditions* (23) *and* (24), *and the equality $J(\tilde{x}(N)) = J(\hat{x}(N))$ holds if and only if the pair $(\hat{x}, \hat{u})$ satisfies the relations of the discrete maximum principle* (20)–(22).

Thus, within the considered problem, it is possible to guarantee the fulfillment of the strict inequality (12) when choosing a new control $\tilde{u}$ from the condition (23) if and only if the given control $\hat{u}$ is not extremal in the problem (14). Directly from this we have the possibility of making an iterative algorithm for approximate construction of an optimal control in the original time-optimization problem.

Let the control $\hat{u} \in \mathcal{U}$ be given. Now we construct a sequence of controls $u^{(l)} \in \mathcal{U}$ in the following way. Let $u^{(0)} = \hat{u}$. Let for some $l \geqslant 0$ the control $u^{(l)}$ have already been constructed. Then for each $k \in \{0, \dots, N-1\}$ we take as $u^{(l+1)}(k)$ the solution to the optimization problem

$$\langle \psi^{(l)}(k+1) - 2(A^{N-k-1})^{\mathrm{T}} A^{N-k} x^{(l+1)}(k), v \rangle - \|A^{N-k-1}v\|^2 \to \max_{v \in U},$$

where $\psi^{(l)}(N) = 0$, $x^{(l+1)}(0) = x_0$ and for $k = 0, \dots, N-1$

$$\psi^{(l)}(k) = A^{\mathrm{T}} \psi^{(l)}(k+1) - 2(A^{N-k})^{\mathrm{T}} A^{N-k-1} u^{(l)}(k),$$
$$x^{(l+1)}(k+1) = Ax^{(l+1)}(k) + u^{(l+1)}(k).$$

From Theorem 3 and Remark 3, taking into account the compactness of the set U and the unique solvability of the last equation, we obtain

Corollary 3. *Let the matrix A be non-singular. Then for any initial approximation $\hat{u} \in \mathcal{U}$ the sequence of control processes $(x^{(l)}, u^{(l)})$ constructed above has a subsequence converging in $\mathbb{R}^{2nN+n}$, and any process $(\tilde{x}, \tilde{u})$ that is a partial limit of the sequence $\{(x^{(l)}, u^{(l)})\}$ satisfies the relations of the discrete maximum principle* (20)–(22).

6. ALGORITHMIC IMPLEMENTATION

In accordance with the obtained results, we have the following algorithm for constructing an approximate solution to the time-optimization problem for the system (1)–(2).

0. Calculate a two-sided estimate of the optimal time $\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}$. In the case when the matrix A has n linearly independent eigenvectors, the desired estimate can be found by Theorem 1 taking into account Remark 1. Put $N = \overline{N_{\min}}$. Specify the values of admissible calculation errors $\varepsilon_1, \varepsilon_2 > 0$.
1. Let $u^{(0)} = 0$, $l = 0$, $\mathcal{A}(k) = A^{N-k}$, $k = 0, \dots, N$.
2. Find the solution $x^{(l)}$ to the system of equations

$$x(k+1) = Ax(k) + u^{(l)}(k), \quad k = 0, \dots, N-1, \quad x(0) = x_0.$$

3. Find the solution $\psi^{(l)}$ to the system of equations

$$\psi(k) = A^{\mathrm{T}} \psi(k+1) - 2\mathcal{A}(k)^{\mathrm{T}} \mathcal{A}(k+1) u^{(l)}(k), \quad k = 0, \dots, N-1, \quad \psi(N) = 0.$$

4. Find sequentially for each $k \in \{0, \dots, N-1\}$ a solution $u^{(l+1)}(k)$ to the optimization problem

$$\langle \psi^{(l)}(k+1) - 2\mathcal{A}(k+1)^{\mathrm{T}} \mathcal{A}(k) x^{(l+1)}(k), v \rangle - \|\mathcal{A}(k+1)v\|^2 \to \max_{v \in U},$$

   where the values of $x^{(l+1)}(k)$ are calculated by formulas

$$x^{(l+1)}(k+1) = Ax^{(l+1)}(k) + u^{(l+1)}(k), \quad k = 0, \dots, N-1, \quad x^{(l+1)}(0) = x_0.$$

5. Check internal stop condition

$$\|x^{(l+1)}(N)\| - \|x^{(l)}(N)\| < \varepsilon_1,$$

   if it is fulfilled, put $\tilde{u} = u^{(l+1)}$ and go to step 7.

6. Increase l by 1 and go to step 3.
7. Check external stop condition

$$\|\tilde{x}(N)\| = \|x^{(l+1)}(N)\| < \varepsilon_2,$$

if it is fulfilled, put $N_{\min} = N$ and finish the calculations, otherwise, if the inequality $N < \underline{N_{\min}}$ holds, then increase N by 1 and go to step 1. If $N = \underline{N_{\min}}$, then finish the calculations by putting $N_{\min} = \underline{N_{\min}}$.

Since the matrix A in the system (1) is non-singular, the algorithm allows for each $N = \overline{N_{\min}}, \ldots, \underline{N_{\min}}$ to approximately find the control that generates a process for which the relations of the discrete maximum principle (20)–(22) are satisfied.

The value of $N_{\min}$, found at the end of the algorithm execution, has the meaning of an upper bound for the optimal time for the system (1)–(2), and the control $\tilde{u}$ should be considered only as a guarantee. However, if in the problem (14) the discrete maximum principle is a necessary and sufficient condition for optimality, then, with an accuracy of up to the ε_2-error of calculations, $N_{\min}$ coincides with the optimal time (i.e., for a sufficiently small ε_2 it cannot differ from the optimal time by more than 1), and $\tilde{u}$ generates a process in the system (1)–(2) that is optimal in terms of optimal time. In the case $N_{\min} = \overline{N_{\min}}$, the value $N_{\min}$ is obviously the optimal time, while $\tilde{u}$ is the optimal control in terms of time-optimization problem.

Steps 1–6 of the algorithm replace a numerical method for solving the problem of minimizing the functional J with respect to the set of variables $u(0), \ldots, u(N-1)$ for a given value of N. Instead of solving one problem with a quadratic functional of the form (11) of dimension nN on the set U^N, it is proposed to sequentially solve N problems of dimension n on the set U with the quadratic functional written down in step 4 several times. Computational practice shows that the second approach is more effective in cases where the value of N is sufficiently large. The complexity of solving the problem in step 4 depends on the structure of the set U. For example, if U is a polyhedron, then the corresponding optimization problem can be solved by the ellipsoid method with polynomial time complexity [32]. We also note that the values of N are quite large if the studied problem was obtained by highly accurate discretization of a continuous-time problem.

7. EXAMPLES

To illustrate the results in this section, we will restrict ourselves to the case $n = 2$ as the most convenient for depicting the trajectories of the controlled process in phase space and at the same time sufficiently meaningful in terms of the diversity of problem statements and the traits of their solutions. We will begin with one academic example that allows almost analytical study.

Example 1. Let's consider a system of the form (1)–(2)

$$x(k+1) = Ax(k) + u(k), \quad x(0) = x_0 \in \mathbb{R}^2,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \quad x_0 = \begin{pmatrix} \sqrt{r^2 - \gamma^2} + r \\ \gamma^{-1} \end{pmatrix}, \quad \gamma \in (0; r), \quad r > 0.$$

The set $U \subset \mathbb{R}^2$, defining the geometric constraints, has the form

$$U = \{(u_1, u_2) \mid u_1^2 + u_2^2 \leqslant r^2\}.$$

We need to solve the time-optimization problem for this system.

It is clear that $N_{\min} \geqslant 2$, since the point

$$Ax_0 = \begin{pmatrix} \sqrt{r^2 - \gamma^2} + r \\ 1 \end{pmatrix}$$

is located at the distance from origin greater than r, which means that it is impossible to transfer the system to the origin in one step.

At the same time, for any γ we can set $u^*(0) = (-r, 0)^{\mathrm{T}}$ and obtain

$$x^*(1) = Ax_0 + u^*(0) = \begin{pmatrix} \sqrt{r^2 - \gamma^2} \\ 1 \end{pmatrix}, \quad Ax^*(1) = \begin{pmatrix} \sqrt{r^2 - \gamma^2} \\ \gamma \end{pmatrix},$$

where the point $Ax^*(1)$ is at a distance r from zero, so that there is $u^*(1)$ for which $x^*(2) = Ax^*(1) + u^*(1) = 0$. Consequently, in this problem $N_{\min} = 2$ and the process $\{x^*(k), u^*(k-1)\}_{k=0}^{2}$ is optimal.

Let us try the algorithm from Section 6 on this example. At the zero step, we define a two-sided estimate $\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}$. Since the matrix A is diagonal and non-singular, then to construct the desired estimate according to Theorem 1 it is sufficient to calculate the values of the four parameters $u'_{1,\max}, u'_{2,\max}, u''_{1,\max}, u''_{2,\max}$ so that the double inclusion is satisfied

$$[-u'_{1,\max}; u'_{1,\max}] \times [-u'_{2,\max}; u'_{2,\max}] \subset U \subset [-u''_{1,\max}; u''_{1,\max}] \times [-u''_{2,\max}; u''_{2,\max}].$$

To perform the first of them, we set

$$u'_{1,\max} = u'_{2,\max} = r/\sqrt{2},$$

and the best values of $u''_{i,\max}$, taking into account the Remark 1, are determined from the solution of the problem (7):

$$u''_{1,\max} = u''_{2,\max} = r.$$

Moreover, due to Theorem 1 we have

$$\left\lceil \max_{i=\overline{1,2}} F(x_{0,i}; r, \lambda_i) \right\rceil \leqslant N_{\min} \leqslant \left\lceil \max_{i=\overline{1,2}} F(x_{0,i}; r/\sqrt{2}, \lambda_i) \right\rceil,$$

where $x_{0,1} = \sqrt{r^2 - \gamma^2} + r$, $x_{0,2} = \gamma^{-1}$, $\lambda_1 = 1$, $\lambda_2 = \gamma$, and the function F is defined by the formula (6). In particular, for any values of $r > 0$ and $\gamma \in (0; r)$

$$N_{\min} \geqslant \lceil F(x_{0,1}; r, \lambda_1) \rceil = \left\lceil \frac{\sqrt{r^2 - \gamma^2} + r}{r} \right\rceil = 2.$$

For definiteness, if $r = 0.5$, $\gamma = 0.1$, then from the upper bound we find

$$N_{\min} \leqslant \lceil 2.8 \rceil = 3.$$

Let $N = 2$, $u^{(0)}(k) \equiv 0$. At step 2 of the algorithm we see that the zero control is not optimal:

$$x^{(0)}(1) = \begin{pmatrix} \sqrt{r^2 - \gamma^2} + r \\ 1 \end{pmatrix}, \quad x^{(0)}(2) = \begin{pmatrix} \sqrt{r^2 - \gamma^2} + r \\ \gamma \end{pmatrix} \quad \Rightarrow \quad \|x^{(0)}(2)\| > 0.$$

At step 3, we fix $\psi^{(0)}(k) \equiv 0$ and go to step 4.

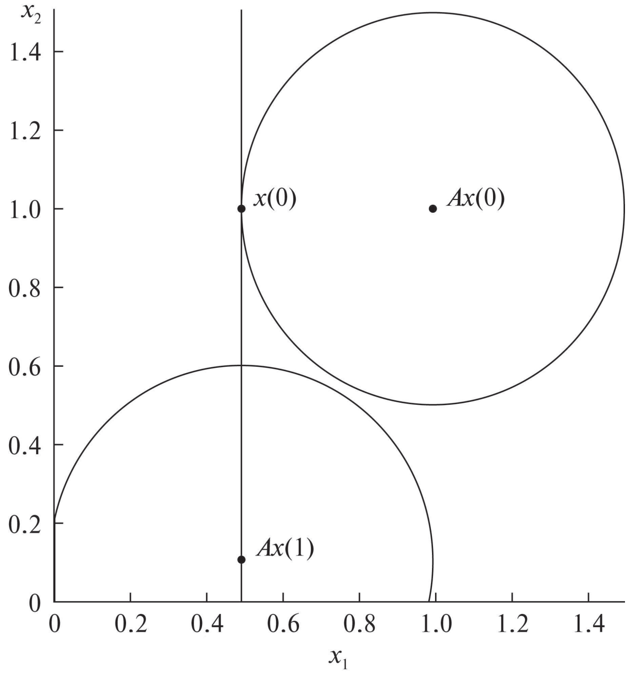

Fig. 1. The first iteration of the algorithm in Example 1.

At step 4, we need to solve two optimization problems sequentially for $k = 0$ and for $k = 1$. In particular, for $k = 0$ we have the problem

$$u_1^2 + \gamma^2 u_2^2 + 2\sqrt{r^2 - \gamma^2}u_1 + 2\gamma^2 u_2 \to \min_{u_1^2+u_2^2 \leqslant r^2}.$$

For $r = 0.5$, $\gamma = 0.1$ from its solution we obtain (see Fig. 1)

$$u^{(1)}(0) \approx (-0.4999, -0.01)^{\mathrm{T}} \quad \Rightarrow \quad x^{(1)}(1) \approx (0.49, 0.99)^{\mathrm{T}}.$$

For the same values of r and γ for $k = 1$ we find

$$u^{(1)}(1) \approx (-0.49, -0.1)^{\mathrm{T}} \quad \Rightarrow \quad x^{(1)}(2) \approx 0.$$

Thus, already at the first iteration of the algorithm, a guaranteeing control in the time-optimization problem for the given system is approximately constructed. Since $N = 2$ coincides with the lower bound of the optimal time, we can conclude that this guaranteeing control is optimal and $N_{\min} = 2$.

Let us make sure that the second iteration does not lead to a worsening of the result. At the second iteration, as the initial control we have the control just constructed $u^{(1)}$. At step 3 we find ($r = 0.5$, $\gamma = 0.1$)

$$\psi^{(1)}(2) = 0, \quad \psi^{(1)}(1) \approx (1, 0.02)^{\mathrm{T}}.$$

At step 4 we obtain

$$u^{(2)}(0) \approx (-0.4992, -0.028)^{\mathrm{T}} \quad \Rightarrow \quad x^{(2)}(1) \approx (0.491, 0.972)^{\mathrm{T}},$$
$$u^{(2)}(1) \approx (-0.4905, -0.97)^{\mathrm{T}} \quad \Rightarrow \quad x^{(2)}(2) \approx 0.$$

It is interesting to note the peculiarities of the geometry of the constructions carried out.

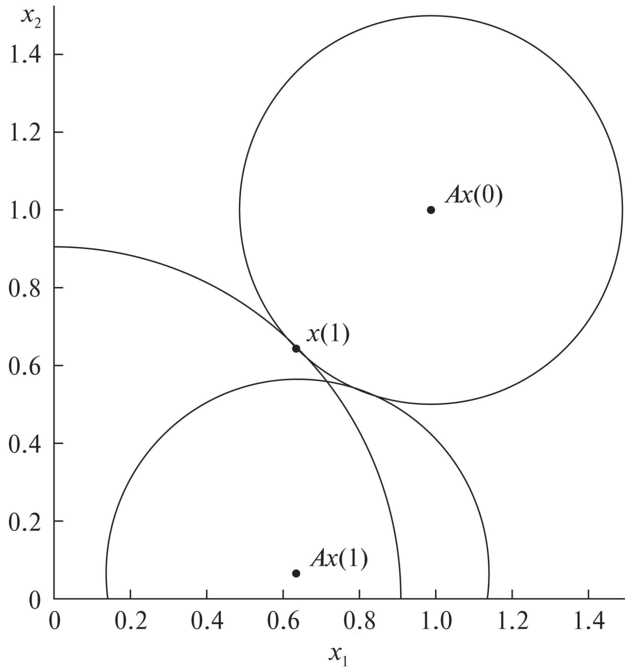

Fig. 2. Search for points closest to origin in Example 1.

To do this, we first assume that the control in the system is constructed so that at each step the closest point to origin from the reachable set is obtained. Since

$$x(1) = \begin{pmatrix} \sqrt{r^2 - \gamma^2} + r \\ 1 \end{pmatrix} + u(0),$$

the corresponding control $u^*(0)$ is the solution to the problem

$$\|x(1)\|^2 = \left(\sqrt{r^2 - \gamma^2} + r + u_1\right)^2 + (1 + u_2)^2 \to \min_{u_1^2 + u_2^2 \leqslant r^2},$$

in particular, for $r = 0.6$, $\gamma = 0.1$ we have

$$u^*(0) \approx (-0.352, -0.355)^{\mathrm{T}} \quad \Rightarrow \quad x^*(1) \approx (0.639, 0.645)^{\mathrm{T}}.$$

But then (see Fig. 2)

$$Ax(1) \approx \begin{pmatrix} 0.639 \\ 0.064 \end{pmatrix},$$

but this point is located at a distance greater than $0.6 > r$ from origin. Therefore, no control $u^*(1)$ will allow the system to be transferred to origin at the second step.

Note now that the first iteration of the algorithm from Section 6 also implements the search for the closest points from the reachable sets, but not in the sense of the Euclidean metric (see Figs. 1 and 2), but in the sense of distances generated by the norms

$$\|x\|^2_{A,N,k} = \|A^{N-k-1}x\|^2.$$

In particular, for $k = N - 1$ the norm $\|\cdot\|_{A,N,k}$ coincides with the Euclidean norm for any N and A, $\det A \neq 0$. The second iteration minimizes this distance not to zero, but to some other point determined by the vector of dual variables $\psi(k)$. This fact allows us to correct the structure of optimal control in the case when the first iteration failed to calculate the answer accurately enough. As will be seen below, such a situation can arise if the set U has a sufficiently complex structure, and the point x_0 lies on the boundary of the null-controllable set $\Xi(N_{\min})$.

Example 2. Let us consider a system of the form (1)–(2), where

$$A = \frac{4}{5}\begin{pmatrix} \cos(1)+\sin(1) & -2\sin(1) \\ \sin(1) & \cos(1)-\sin(1) \end{pmatrix}, \quad x_0 = \begin{pmatrix} -37.8 \\ -26.1 \end{pmatrix}.$$

Geometric constraints are defined by a set

$$U = \{u \mid \langle u, Hu\rangle \leqslant 1\}, \quad H = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Consider the time-optimization problem.

Let us construct an estimate $\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}$. The matrix A has a pair of complex eigenvalues, which are equal in absolute value to $r_1 = 4/5$. The transition matrix to the real Jordan basis has the form

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the considered system satisfies the conditions of Theorem 1 for the case $n_1 = 0$, $n_2 = 1$. The radii of the inscribed $\underline{U}$ and circumscribed $\overline{U}$ balls for the ellipse $S^{-1}U$ are determined uniquely and can be found numerically:

$$\underline{U} = \mathcal{B}_{R'_{1,\max}}, \quad R'_{1,\max} = 0.3449, \quad \overline{U} = \mathcal{B}_{R''_{1,\max}}, \quad R''_{1,\max} = 1.2965.$$

Using the values $R'_{1,\max}$, $R''_{1,\max}$ in Theorem 1, we obtain that

$$8 \leqslant N_{\min} \leqslant 13.$$

We apply the algorithm from the Section 6, performing the corresponding calculations numerically. We fix the found two-sided estimate $8 \leqslant N_{\min} \leqslant 13$ and set $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 10^{-16}$. With these error values for $N = 8$ and $N = 9$, the internal stopping condition is satisfied at the 33th and 9th iterations, respectively, but the external stopping condition is satisfied only for $N = 10$. For $N = 10$, the internal stopping condition is satisfied already at the first iteration. In this case, the control process $\{x^{(1)}(k), u^{(1)}(k-1)\}_{k=0}^{10}$ shown in Fig. 3 is found (in this and subsequent Figs. 4, 6–8 and 10, the trajectory of the process in the phase space is shown on the left, and the values of the controls against the background of a set of geometric constraints U are shown on the right). For this process, it holds that $\|x^{(1)}(N)\| < \varepsilon_2$. For comparison, for $N = 8$, $\|x^{(33)}(N)\| \approx 0.5$ was found, and for $N = 9$, $\|x^{(9)}(N)\| \approx 0.2$ was found. Note that calculations up to the internal stop for $N = 11, 12, 13$ in one iteration lead to constructing processes for which the external stop condition $\|x^{(1)}(N)\| < \varepsilon_2$ holds, and in this case $u^{(1)}(10) \approx \ldots \approx u^{(1)}(N) \approx 0$.

The formal result of applying the algorithm from Section 6 is the following: with an accuracy of up to ε_2-error of calculations, the upper bound of the optimal time $N_{\min}$ can be reduced from $\underline{N_{\min}} = 13$ to $\underline{N_{\min}} = 10$, while the guaranteeing control and the corresponding trajectory have the form, shown in Fig. 3.

To evaluate the quality of the obtained results, we can use numerical procedures of conditional minimization for direct solution of smooth finite-dimensional problems $\|x(9)\|^2 \to \min$ and $\|x(10)\|^2 \to \min$ in the presence of a finite number of smooth inequality-type constraints on the

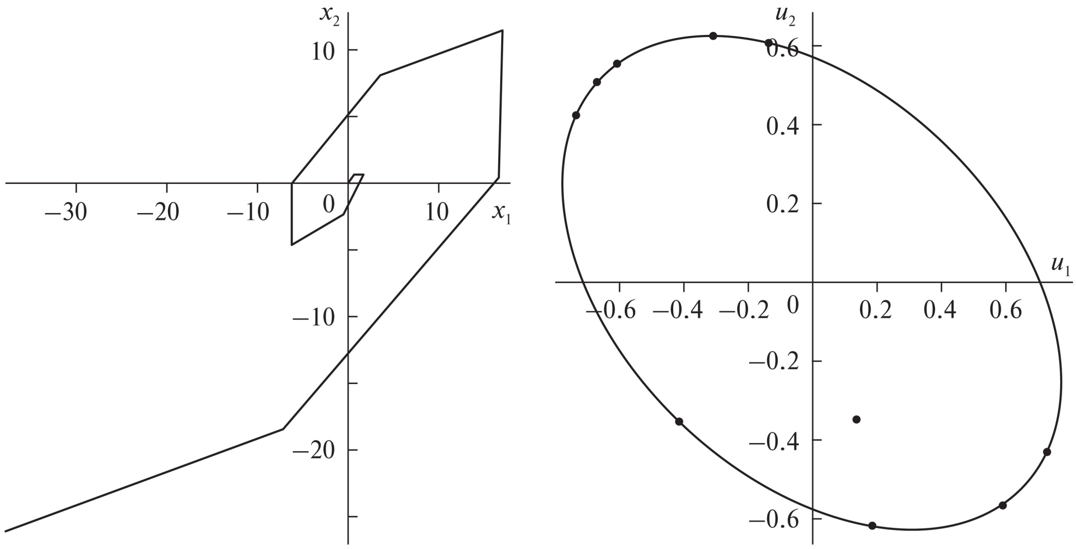

Fig. 3. Guaranteeing (optimal) process in Example 2.

variables $u(k)$. Since the dimensions of the corresponding problems are not too large, they can be solved quite accurately. In the first case, the minimum value is approximately equal to $0.041 \approx 0.2^2$, in the second case it is equal to zero. Consequently, $N_{\min} = 10$ and the guaranteeing process shown in Fig. 3 is optimal.

Example 3. Let us consider a system of the form (1)–(2), where

$$A = \begin{pmatrix} 31/20 & -3/20 \\ 1/10 & 6/5 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 5.08 \\ 6.28 \end{pmatrix}.$$

Geometric constraints are defined by a set

$$U = \left\{ (u_1, u_2) \;\middle|\; \frac{4^{2/3}|u_1 - \sqrt{3}u_2|^{4/3}}{16} + \frac{6^{2/3}|\sqrt{3}u_1 + u_2|^{4/3}}{36} \leqslant 1 \right\}.$$

Consider the time-optimization problem.

Let us construct an estimate $\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}$. The matrix A has a pair of real eigenvalues $\lambda_1 = 3/2$, $\lambda_2 = 5/4$. The transition matrix to the real Jordan basis has the form

$$S = \begin{pmatrix} 3 & 1/2 \\ 1 & 1 \end{pmatrix}.$$

The considered system satisfies the conditions of Theorem 1 for the case $n_1 = 2$, $n_2 = 0$. The parameters of the inscribed $\underline{U}$ and circumscribed $\overline{U}$ rectangles for the set $S^{-1}U$ are determined numerically:

$$\underline{U} = [-u'_{1,\max}; u'_{1,\max}] \times [-u'_{2,\max}; u'_{2,\max}], \quad u'_{1,\max} = 0.3883, \quad u'_{2,\max} = 1.4057,$$
$$\overline{U} = [-u''_{1,\max}; u''_{1,\max}] \times [-u''_{2,\max}; u''_{2,\max}], \quad u''_{1,\max} = 0.8834, \quad u''_{2,\max} = 2.4839.$$

Using Theorem 1, we obtain the estimate

$$3 \leqslant N_{\min} \leqslant 17.$$

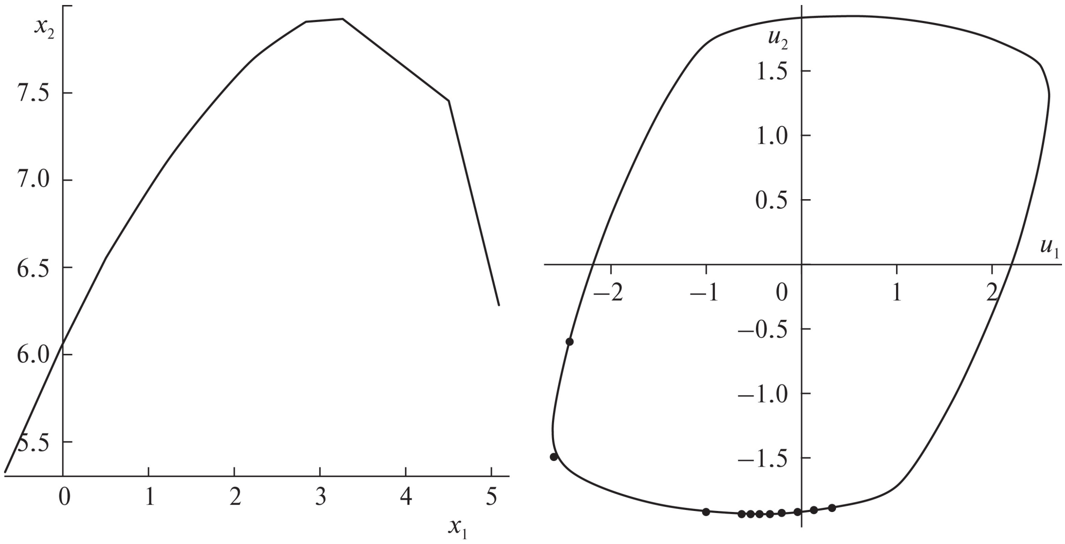

Fig. 4. Process $(x^{(1)}, u^{(1)})$ in Example 3 for $N = 11$.

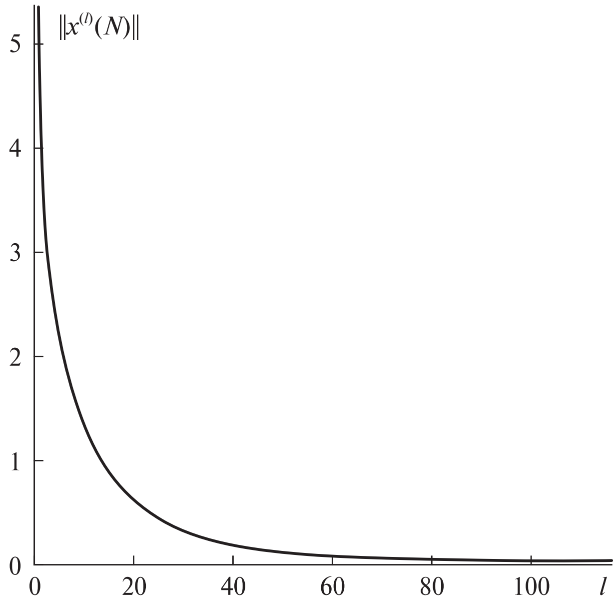

Fig. 5. Convergence of inner iterations in Example 3 for $N = 11$.

Let us numerically apply the algorithm from Section 6 for $\varepsilon_1 = 10^{-4}$ and $\varepsilon_2 = 0.001$. The external stopping condition is first satisfied for $N = 11$. Let us describe the process of executing internal iterations for $N = 11$ in more detail. At the first internal iteration, we find the control process $\{x^{(1)}(k), u^{(1)}(k-1)\}_{k=0}^{11}$, shown in Fig. 4. For this process $\|x^{(1)}(N)\| \approx 5.37$. The use of repeated iterations $(l = 1, 2, \ldots)$ leads to a decrease in the value of $\|x^{(l+1)}(N)\|$. The graph of the dependence of $\|x^{(l)}(N)\|$ on l is shown in Fig. 5. The control process $(x^{(116)}, u^{(116)})$ is shown in Fig. 6.

The low accuracy of the two-sided estimate and the slow convergence of the algorithm in this example are due to the fact that the system under consideration is not stable, and the initial condition lies near the boundary of the reachable set. Nevertheless, based on the results of the algorithm's operation, a new upper bound for the optimal time was obtained with the ε_2-error:

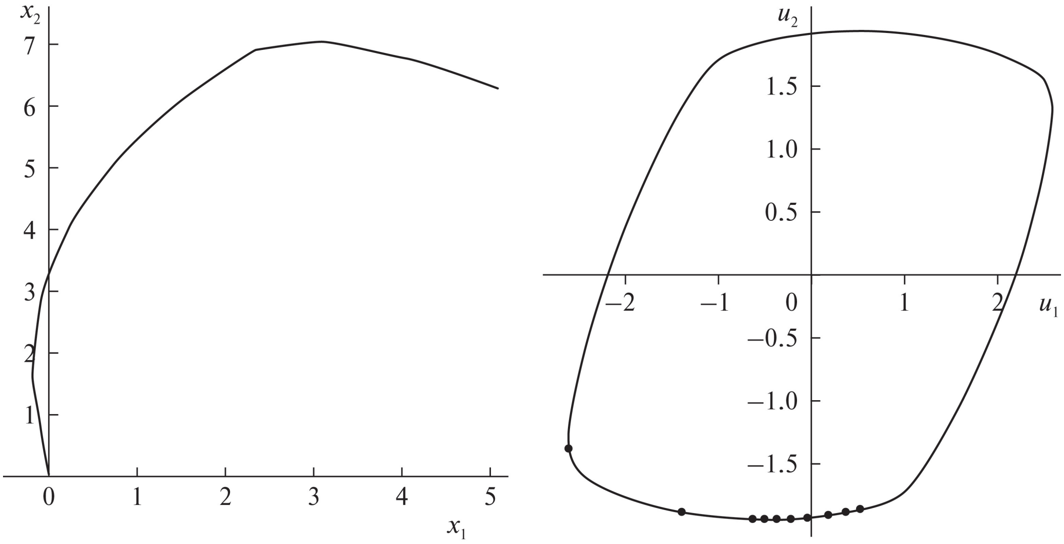

Fig. 6. Guarantee process in Example 3.

$\underline{N_{\min}} = 11$, and a guaranteeing control corresponding to this value was constructed. However, this bound does not coincide with the optimal time, since the minimum value of the objective functional in the finite-dimensional problem $\|x(10)\|^2 \to \min$ is equal to zero, and in reality $N_{\min} = 10$.

Example 4. Let us consider a system of the form (1)–(2), where

$$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 9.33 \\ 0.2 \end{pmatrix}.$$

In this example, we consider geometric constraints of mixed type defined by the set

$$U = \left[-\frac{\sqrt{3}}{2}; \frac{\sqrt{3}}{2}\right]^2 \cap \mathcal{B}_1,$$

where $\mathcal{B}_1$ is the unit ball with center at origin. Let us solve the time-optimization problem under these conditions.

Let us construct an estimate $\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}$. The matrix A has a pair of complex conjugate eigenvalues equal in absolute value to $r_1 = 1$. At the same time, it is already in its real Jordan form, and therefore $S = I$. Thus, the system under consideration satisfies the conditions of Theorem 1 for the case $n_1 = 0$, $n_2 = 1$. The radii of the inscribed $\underline{U}$ and circumscribed $\overline{U}$ balls for the set $S^{-1}U$ are uniquely determined and can be found numerically:

$$\underline{U} = \mathcal{B}_{R'_{1,\max}}, \quad R'_{1,\max} = \sqrt{3}/2, \quad \overline{U} = \mathcal{B}_{R''_{1,\max}}, \quad R''_{1,\max} = 1.$$

Using Theorem 1, we obtain the estimate

$$10 \leqslant N_{\min} \leqslant 11.$$

We apply the algorithm from Section 6 for $\varepsilon_1 = \varepsilon_2 = 10^{-4}$. For $N = 10$, at the first internal iteration we find the control process $\{x^{(1)}(k), u^{(1)}(k-1)\}_{k=0}^{10}$, shown in Fig. 7, for which the condition $\|x^{(1)}(N)\| < \varepsilon_2$ holds.

Since $N = 10$ is the lower bound on the optimal time, the process, shown in Fig. 7, is optimal to within ε_2-calculation error.

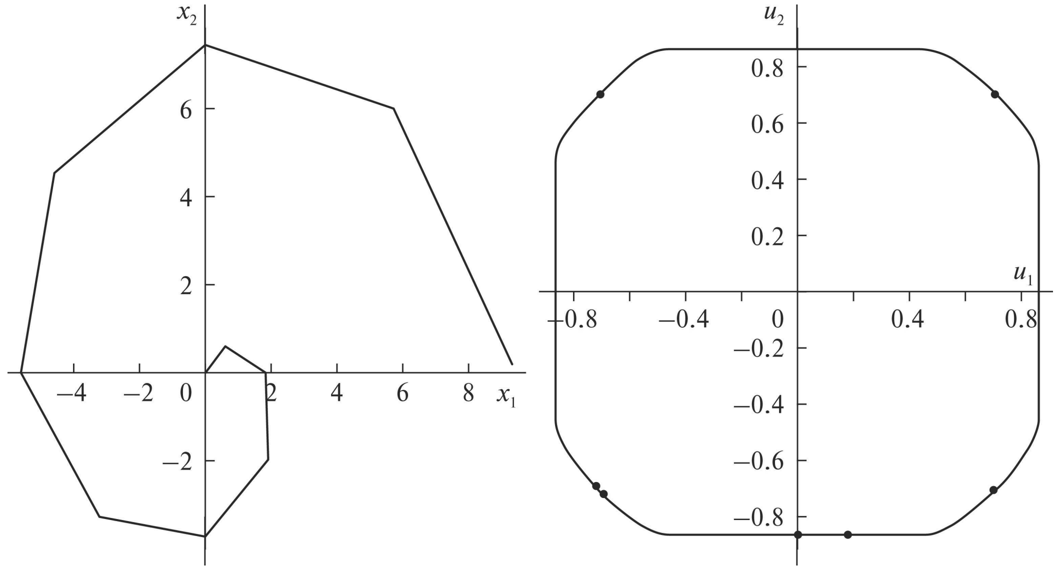

Fig. 7. Optimal process in Example 4.

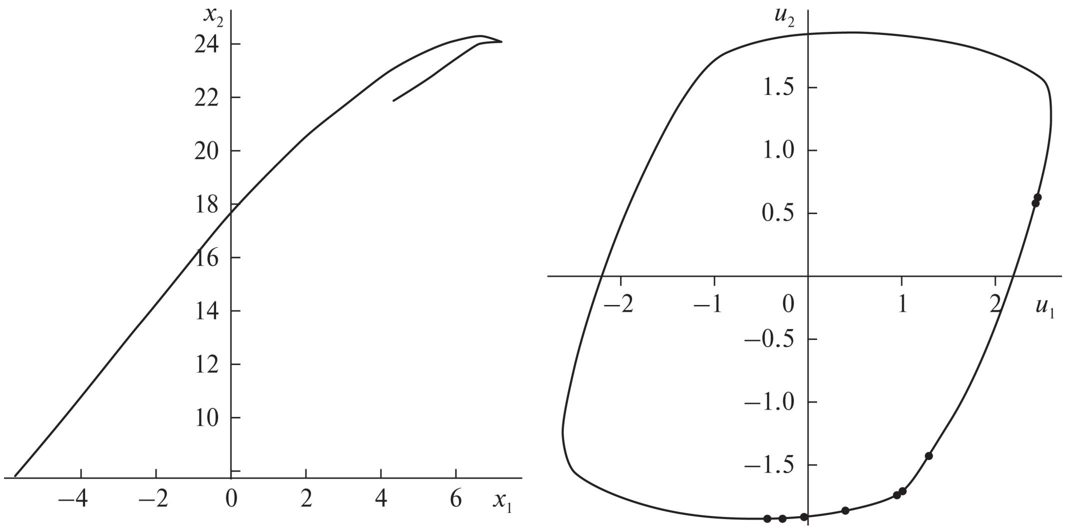

Fig. 8. The process $(x^{(1)}, u^{(1)})$ in Example 5.

Example 5. Let us consider a system of the form (1)–(2), where

$$A = \begin{pmatrix} 33/20 & -1/5 \\ 4/5 & 17/20 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 4.31 \\ 21.85 \end{pmatrix}.$$

The geometric constraints are defined by the same set as in Example 3. Let us consider the time-optimization problem for this example.

The trait of this example is that the matrix A does not have two linearly independent eigenvectors. Therefore, Theorem 1 cannot be used here to construct estimates of the optimal time. However, it is possible to use the means for determining the optimal time developed in [22]. Due to [22], the exact equality $N_{\min} = 10$ holds.

We apply the algorithm from the Section 6, considering the two-sided estimate of the optimal time $N_{\min}$ to be given in the form $10 \leqslant N_{\min} \leqslant 10$ and setting $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 0.01$. At the first internal iteration, we find the control process $\{x^{(1)}(k), u^{(1)}(k-1)\}_{k=0}^{10}$, shown in Fig. 8. As in

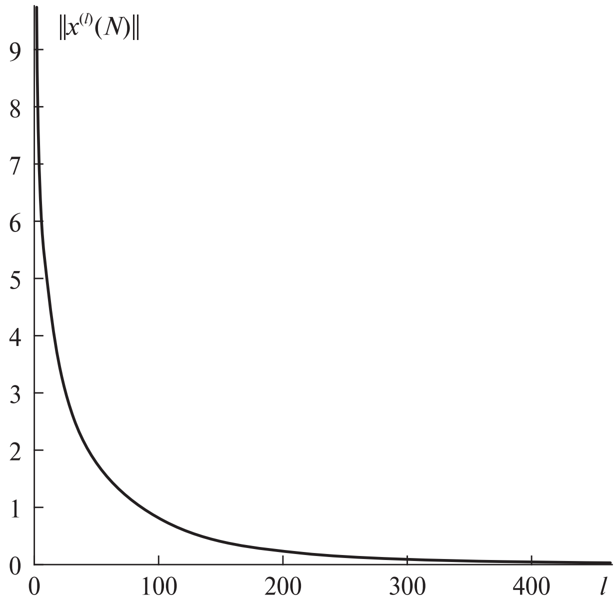

Fig. 9. Convergence of iterations to the optimal solution in Example 5.

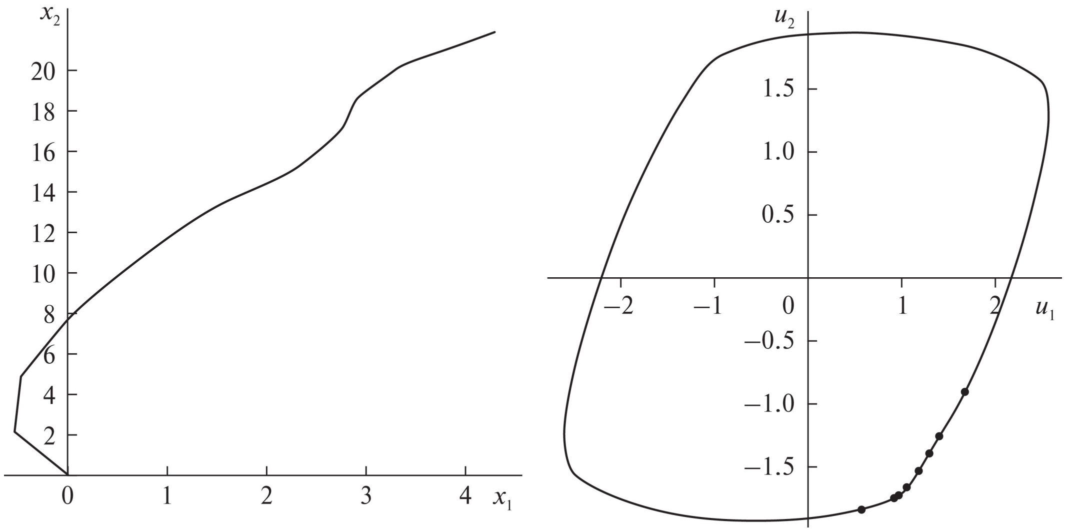

Fig. 10. Approximation to the optimal process in Example 5.

Example 3, this process is far from optimal, since $\|x^{(1)}(N)\| \approx 9.73$. Therefore, the use of repeated iterations is relevant, which leads to the results shown in Figs. 9 and 10.

Since the optimal time is known exactly, the obtained result is a ε_2-approximation to the optimal solution to the problem.

8. CONCLUSION

The paper constructs an algorithm for solving the time-optimization problem for linear discrete-time systems with a non-singular matrix. The algorithm allows us to refine known upper estimates for the optimal time and find control processes that guarantee the corresponding estimates. When some additional assumptions are fulfilled, the result of the algorithm is the construction of an optimal solution to the time-optimization problem.

The proposed algorithm belongs to the class of methods for solving finite-dimensional optimization problems with constraints based on the dimensionality reduction procedure. Since the average dimensionality reduction coefficient in the considered problem is determined by the optimal time, the efficiency of the algorithm increases with its increase.

In the future, we are planning to extend the obtained results to the case of systems with a singular matrix, and also to use the proposed approach for direct calculation of the optimal time, and not only its upper bound. It is also relevant to obtain new meaningful statements about convergence and determine the convergence rate of the algorithm in the general case.

FUNDING

Theorem 1 was proved by D.N. Ibragimov using funds from the Russian Science Foundation project no. 25-21-00018 https://rscf.ru/project/25-21-00018 in MAI (NIU). Theorems 2 and 3 was proved by K.A. Tsarkov using funds from the Russian Science Foundation project no. 22-11-00042 https://rscf.ru/project/22-11-00042 in ICS RAS.

APPENDIX

Lemma 5. *Let* $V_1, V_1' \subset \mathbb{R}^{n_1}$, $V_2, V_2' \subset \mathbb{R}^{n_2}$. *Then*

$$V_1 \times V_2 + V_1' \times V_2' = (V_1 + V_1') \times (V_2 + V_2').$$

Proof. The inclusion $y \in V_1 \times V_2 + V_1' \times V_2'$ by the definition of the Minkowski sum and the Cartesian product is valid if and only if there exist $v_1 \in V_1$, $v_2 \in V_2$, $v_1' \in V_1'$, $v_2' \in V_2'$ such that

$$y = (v_1, v_2) + (v_1', v_2') = (v_1 + v_1', v_2 + v_2').$$

But the last condition is equivalent to the inclusion $y \in (V_1 + V_1') \times (V_2 + V_2')$. Lemma 5 is proved.

Proof of Lemma 2. According to the assumptions $\det A \neq 0$ and for any $k \in \mathbb{N}$ the following relations hold:

$$A^{-k} = \begin{pmatrix} \lambda_1^{-k} & & & \ldots & & 0 \\ & \ddots & & & & \\ & & \lambda_{n_1}^{-k} & & & \\ \vdots & & & r_1^{-k} A_{-k\varphi_1} & & \vdots \\ & & & & \ddots & \\ 0 & & & \ldots & & r_{n_2}^{-k} A_{-k\varphi_{n_2}} \end{pmatrix},$$

$$A^{-k}U = \bigotimes_{i=1}^{n_1} \Big[-|\lambda_i|^{-k} u_{i,\max}; |\lambda_i|^{-k} u_{i,\max} \Big] \times \bigotimes_{j=1}^{n_2} r_j^{-k} \mathcal{B}_{R_{j,\max}}.$$

Therefore, from the definition of (4) and Lemmas 1, 5 for an arbitrary $N \in \mathbb{N}$ follows the representation

$$\Xi(N) = \bigotimes_{i=1}^{n_1} \left[-\sum_{k=1}^{N} |\lambda_i|^{-k} u_{i,\max}; \sum_{k=1}^{N} |\lambda_i|^{-k} u_{i,\max} \right] \times \bigotimes_{j=1}^{n_2} \sum_{k=1}^{N} r_j^{-k} \mathcal{B}_{R_{j,\max}}. \tag{A.1}$$

Hence, the inclusion $x_0 \in \Xi(N)$ is equivalent to the fact that for all $i = \overline{1, n_1}$ and $j = \overline{1, n_2}$ the following relations are satisfied:

$$|x_{0,i}| \leqslant \sum_{k=1}^{N} |\lambda_i|^{-k} u_{i,\max} = \begin{cases} N u_{i,\max}, & |\lambda_i| = 1, \\ u_{i,\max} \dfrac{1 - |\lambda_i|^{-N}}{|\lambda_i| - 1}, & |\lambda_i| \neq 1, \end{cases} \tag{A.2}$$

$$\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2} \leqslant \sum_{k=1}^{N} r_j^{-k} R_{j,\max} = \begin{cases} N R_{j,\max}, & r_j = 1, \\ R_{j,\max} \dfrac{1 - r_j^{-N}}{r_j - 1}, & r_j \neq 1. \end{cases} \tag{A.3}$$

The condition (A.2) is equivalent to the inequality

$$N \geqslant \begin{cases} \dfrac{|x_{0,i}|}{u_{i,\max}}, & |\lambda_i| = 1, \\ -\dfrac{\ln\left(1 - \frac{|x_{0,i}|}{u_{i,\max}}(|\lambda_i| - 1)\right)}{\ln|\lambda_i|}, & |\lambda_i| \neq 1, \end{cases} = F(x_{0,i}; u_{i,\max}, \lambda_i).$$

The condition (A.3) is equivalent to the inequality

$$N \geqslant \begin{cases} \dfrac{\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2}}{R_{j,\max}}, & r_j = 1, \\ -\dfrac{\ln\left(1 - \frac{\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2}}{R_{j,\max}}(r_j - 1)\right)}{\ln r_j}, & r_j \neq 1, \end{cases}$$
$$= F\left(\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2}; R_{j,\max}, r_j\right).$$

The resulting expressions are correct, since the following condition is valid:

$$\left(1 - \frac{|x_{0,i}|}{u_{i,\max}}(|\lambda_i| - 1)\right) > 0, \quad i = \overline{1, n_1},$$
$$\left(1 - \frac{\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2}}{R_{j,\max}}(r_j - 1)\right) > 0, \quad j = \overline{1, n_2}.$$

Indeed, for $|\lambda_i| \leqslant 1$ and $r_j \leqslant 1$ these relations are satisfied automatically. If $|\lambda_i| > 1$ or $r_j > 1$, then they can be obtained by passing in (A.1) to the limit with respect to $N \to \infty$ and using the assumption $N_{\min} < \infty$, due to which it holds that $x_0 \in \cup_{N=0}^{\infty} \Xi(N)$ (see the detailed justification in [27]).

Hence, conditions (A.2) and (A.3) are satisfied exactly when

$$N \geqslant \max\left\{ \max_{i=\overline{1,n_1}} F(x_{0,i}; u_{i,\max}, \lambda_i); \max_{j=\overline{1,n_2}} F\left(\sqrt{x_{0,n_1+2j-1}^2 + x_{0,n_1+2j}^2}; R_{j,\max}, r_j\right) \right\},$$

and thus Lemma 2 is proved.

Proof of Lemma 3. It follows from (4) that for any $N \in \mathbb{N} \cup \{0\}$ the inclusion

$$\underline{\Xi}(N) \subset \Xi(N) \subset \overline{\Xi}(N)$$

holds, where $\underline{\Xi}(N), \Xi(N), \overline{\Xi}(N)$ are the null-controllable sets in N steps of the systems $(A, \underline{U})$, (A, U), $(A, \overline{U})$, respectively. Therefore, the inequality $\overline{N_{\min}} \leqslant N_{\min} \leqslant \underline{N_{\min}}$ follows directly from (5). Lemma 3 is proved.

Proof of Theorem 1. Let us denote by $\{\tilde{\Xi}(N)\}_{N=0}^{\infty}$ the class of null-controllable sets of the system $(\Lambda, S^{-1}U)$. According to Lemma 4 the following representation holds:

$$\Xi(N) = S\tilde{\Xi}(N).$$

Hence, the inclusion $x_0 \in \Xi(N)$ holds if and only if $y_0 = S^{-1}x_0 \in \tilde{\Xi}(N)$ holds. Thus, taking into account (5), the value of the optimal time for systems (A, U) and $(\Lambda, S^{-1}U)$ coincides for the initial states x_0 and y_0, respectively.

Let

$$\underline{U} = \bigotimes_{i=1}^{n_1} [-u'_{i,\max}; u'_{i,\max}] \times \bigotimes_{j=1}^{n_2} \mathcal{B}_{R'_{j,\max}}, \quad \overline{U} = \bigotimes_{i=1}^{n_1} [-u''_{i,\max}; u''_{i,\max}] \times \bigotimes_{j=1}^{n_2} \mathcal{B}_{R''_{j,\max}}.$$

Since $0 \in \mathrm{int} U$, and the set U is bounded, the same holds for the set $S^{-1}U$. Consequently, there are values $u'_{i,max}, u''_{i,max}, R'_{j,max}, R''_{j,max} > 0$ for which $\underline{U} \subset S^{-1}U \subset \overline{U}$ holds.

Further, due to Corollary 1 the value of the optimal time $\underline{N_{\min}}$ for the system $(\Lambda, \underline{U})$ and the initial state y_0 has the form

$$\underline{N_{\min}} = \left\lceil \max\left\{ \max_{i=\overline{1,n_1}} F(y_{0,i}; u'_{i,\max}, \lambda_i); \max_{j=\overline{1,n_2}} F\left(\sqrt{y^2_{0,n_1+2j-1} + y^2_{0,n_1+2j}}; R'_{j,\max}, r_j\right)\right\}\right\rceil.$$

Similarly, the value of the optimal time $\overline{N_{\min}}$ for the system $(\Lambda, \overline{U})$ and the initial state y_0 has the form

$$\overline{N_{\min}} = \left\lceil \max\left\{ \max_{i=\overline{1,n_1}} F(y_{0,i}; u''_{i,\max}, \lambda_i); \max_{j=\overline{1,n_2}} F\left(\sqrt{y^2_{0,n_1+2j-1} + y^2_{0,n_1+2j}}; R''_{j,\max}, r_j\right)\right\}\right\rceil.$$

To complete the proof, it remains to apply Lemma 3.

Lemma 6. *Let $\hat{u} \in \mathcal{U}$, $\hat{x} \in \mathcal{X}$ be the solution to* (21)*, $\hat{\psi} \in \mathcal{X}'$ is the solution to* (22)*. Then*

$$\hat{R}(k, \hat{x}(k), \hat{u}(k)) = \min_{x \in \mathbb{R}^n} \hat{R}(k, x, \hat{u}(k)) \quad \forall k \in \{0, \ldots, N-1\}, \tag{A.4}$$

$$\hat{G}(\hat{x}(N)) = \max_{x \in \mathbb{R}^n} \hat{G}(x). \tag{A.5}$$

Proof. The equality (A.5) is obviously satisfied, since

$$\hat{G}(x) \equiv -\langle \hat{\psi}(0), x_0 \rangle + \|A^N x_0\|^2.$$

Due to (22) for all $k \in \{0, \ldots, N-1\}$ we have

$$\hat{R}(k, x, \hat{u}(k)) = \langle \hat{\psi}(k+1), Ax + \hat{u}(k) \rangle - \|A^{N-k-1}(Ax + \hat{u}(k))\|^2$$
$$- \langle \hat{\psi}(k), x \rangle + \|A^{N-k}x\|^2 = \langle \hat{\psi}(k+1), \hat{u}(k) \rangle - \|A^{N-k-1}\hat{u}(k)\|^2,$$

therefore (A.4) also holds. Lemma 6 is proved.

Proof of Theorem 2. Let all the conditions listed in Theorem 2 be satisfied. Based on the notations, introduced in Section 5, due to Lemma 6 we have

$$J(\tilde{x}(N)) = \hat{G}(\tilde{x}(N)) - \hat{\varphi}(N, \tilde{x}(N)) + \hat{\varphi}(0, x_0)$$
$$= \hat{G}(\tilde{x}(N)) - \sum_{k=0}^{N-1} \Big(\hat{\varphi}(k+1, \tilde{x}(k+1)) - \hat{\varphi}(k, \tilde{x}(k))\Big) = \hat{G}(\tilde{x}(N)) - \sum_{k=0}^{N-1} \hat{R}(k, \tilde{x}(k), \tilde{u}(k))$$
$$\overset{(23)}{\leqslant} \hat{G}(\tilde{x}(N)) - \sum_{k=0}^{N-1} \hat{R}(k, \tilde{x}(k), \hat{u}(k)) \overset{(A.4)}{\leqslant} \hat{G}(\tilde{x}(N)) - \sum_{k=0}^{N-1} \hat{R}(k, \hat{x}(k), \hat{u}(k))$$
$$\overset{(A.5)}{\leqslant} \hat{G}(\hat{x}(N)) - \sum_{k=0}^{N-1} \hat{R}(k, \hat{x}(k), \hat{u}(k)) = J(\hat{x}(N)).$$

Theorem 2 is proved.

Proof of Theorem 3. Let us show that in the case of a non-singular matrix A there is a unique pair $(\tilde{x}, \tilde{u})$ satisfying (23), (24). Since due to (24) $\tilde{x}$ is uniquely determined by $\tilde{u}$, it is suffices to show that for any values of $\tilde{x}(k)$ there is a unique $\tilde{u} \in \mathcal{U}$ satisfying the condition (23). Indeed, if this is so, then $\tilde{u}(0) \in U$ is uniquely determined for $k = 0$, by which $\tilde{x}(1)$ is uniquely determined, then $\tilde{u}(1)$, etc. But $\tilde{u}$ from the condition (23) is found for each $k = 0, \ldots, N$ by solving the optimization problem

$$\hat{R}(k, \tilde{x}(k), v) \to \max_{v \in U},$$

which, due to Remark 3, is equivalent to the problem

$$f(v) := \langle \hat{\psi}(k+1) - 2(A^{N-k-1})^{\mathrm{T}} A^{N-k} \tilde{x}(k), v\rangle - \|A^{N-k-1} v\|^2 \to \max_{v \in U}.$$

Now we show that the solution to this problem exists and is unique for any $\tilde{x}(k) \in \mathbb{R}^n$. Since the matrix of the quadratic form $(A^{N-k-1})^{\mathrm{T}} A^{N-k-1}$ is positive definite for $\det A \neq 0$, then the strict global maximum of the function $f : \mathbb{R}^n \to \mathbb{R}$ in the absence of the constraint $v \in U$ is achieved at the point

$$v^* = \frac{1}{2} A^{k+1-N} (A^{k+1-N})^{\mathrm{T}} \hat{\psi}(k+1) - A\tilde{x}(k).$$

There are two possible cases: $v^* \in U$ or $v^* \notin U$. In the first case, v^* is the unique solution to the considered optimization problem with constraints. In the second case, the conditional maximum is attained at some other point $v' \in U$, since the function f is continuous and the set U is compact. Let $\alpha = f(v') = \max_{v \in U} f(v)$. Then the level set $V = \{v \in \mathbb{R}^n \mid f(v) \geqslant \alpha\}$ is a nonempty strictly convex compact set. Moreover, $V \cap U = \{v'\}$. Indeed, by definition $v' \in V \cap U$, and if there were a point $v'' \in V \cap U$, $v'' \neq v'$, then the segment $[v', v'']$ would be entirely contained in the set $V \cap U$, since the sets V and U are convex. But the set V is strictly convex, therefore $\frac{1}{2}(v' + v'') \in \mathrm{int} V$, i.e. $f(v'/2 + v''/2) > \alpha$, $v'/2 + v''/2 \in U$, which contradicts the definition of the number α. Thus, in the second case, v' is the unique solution to the problem $f(v) \to \max_{v \in U}$.

Suppose that the pair $(\hat{x}, \hat{u})$ does not satisfy the relations of the discrete maximum principle (20)–(22). By Remark 2, this means that there is $r \in \{0, \ldots, N-1\}$ such that

$$\hat{R}(r, \hat{x}(r), \hat{u}(r)) < \max_{v \in U} \hat{R}(r, \hat{x}(r), v).$$

Let us take the smallest such r. Then, due to the unique definition of $\tilde{u} \in \mathcal{U}$, we have

$$\tilde{u}(k) = \hat{u}(k), \quad k = 0, 1, \ldots, r-1,$$

and, therefore, $\tilde{x}(r) = \hat{x}(r)$. Hence,

$$\hat{R}(r, \hat{x}(r), \hat{u}(r)) < \max_{v \in U} \hat{R}(r, \hat{x}(r), v) = \max_{v \in U} \hat{R}(r, \tilde{x}(r), v) = \hat{R}(r, \tilde{x}(r), \tilde{u}(r)).$$

From here, returning to the proof of Theorem 2, we find that

$$\sum_{k=0}^{N-1} \hat{R}(k, \tilde{x}(k), \tilde{u}(k)) = \sum_{k=0}^{r-1} \hat{R}(k, \tilde{x}(k), \tilde{u}(k)) + \hat{R}(r, \tilde{x}(r), \tilde{u}(r))$$
$$+ \sum_{k=r+1}^{N-1} \hat{R}(k, \tilde{x}(k), \tilde{u}(k)) > \sum_{k=0}^{N-1} \hat{R}(k, \hat{x}(k), \hat{u}(k))$$

and, therefore,

$$J(\tilde{x}(N)) < J(\hat{x}(N)).$$

If the pair $(\hat{x}, \hat{u})$ satisfies the relations of the discrete maximum principle, then, due to the unique solvability of conditions (23), (24), $\tilde{x} = \hat{x}$ and $\tilde{u} = \hat{u}$. Therefore, $J(\tilde{x}(N)) = J(\hat{x}(N))$. Theorem 3 is completely proved.

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This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board